THE TEMPERATURE FIELD IN A CRYSTAL PLATE WITH A RECTANGULAR NOTCH

Yu. M. Kolyano, I. I. Verba, and I. T. Goryn'

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We have used the functional sweeping method to find a solution for the nonlinear problem of the conduction of heat in the case of an infinite plate with a rectangular notch. Additionally, we have studied the temperature field.

Let us examine an infinite isotropic crystal plate out of which a rectangular section with dimensions $2a_1 \times 2a_2$ has been cut. The plate is heated over the $x_3 = \pm\delta$ surfaces in accordance with the Stefan–Boltzmann law. A heat flow q_0 is specified for the rectangular boundaries of the plate. The thermophysical characteristics of the plate are dependent on temperature. To determine the stationary temperature field in the plate, we have available to us the following boundary-value problem:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[\lambda(t)\frac{\partial t}{\partial x_i}\right] = 0, \tag{1}$$

$$\lambda(t)\frac{\partial t}{\partial x_3} \pm (\sigma_3 t^4 - q_c) = 0 \text{ when } x_3 = \pm\delta, \tag{2}$$

$$\lambda(t)\frac{\partial t}{\partial x_i} = \mp q_0 \text{ when } x_i = \pm a_i,\ |x_{i\pm1}| < a_{i\pm1}, \tag{3}$$

$$t|_{|x_i|\to\infty} < \infty,\ \left.\frac{\partial t}{\partial x_3}\right|_{|x_i|\to\infty} = 0 \quad (i = 1, 2), \tag{4}$$

where

$$i \pm 1 = \begin{cases} 2, & i = 1, \\ 1, & i = 2. \end{cases}$$

For many nonmetallic crystals at temperatures below the Debye temperature the coefficient of thermal conductivity is proportional to the cube of the temperature [1]:

$$\lambda(t) = \varkappa t^3. \tag{5}$$

In this case the boundary-value problem (1)-(4) is completely linearized by means of the Kirchhoff variable

$$\vartheta = \frac{1}{\varkappa}\int_0^t \lambda(t)\,dt \tag{6}$$

which is then brought to the form

$$\frac{\partial^2\vartheta}{\partial x_1^2} + \frac{\partial^2\vartheta}{\partial x_2^2} + \frac{\partial^2\vartheta}{\partial x_3^2} = 0, \tag{7}$$

Institute of Applied Problems in Mechanics and Mathematics, Academy of Sciences of the Ukrainian SSR, L'vov. I. Ya. Franko State University, L'vov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 57, No. 4, pp. 674-680, October, 1989. Original article submitted February 12, 1987.

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$$\varkappa \frac{\partial \vartheta}{\partial x_3} \pm (4\sigma_3\vartheta - q_c) = 0 \quad \text{when } x_3 = \pm \delta, \tag{8}$$

$$\varkappa \frac{\partial \vartheta}{\partial x_i} = \mp q_0 \text{ when } x_i = \pm a_i, \ |x_{i\pm 1}| < a_{i\pm 1} \quad (i = 1, 2), \tag{9}$$

$$\vartheta|_{|x_i| \to \infty} < \infty, \ \left.\frac{\partial \vartheta}{\partial x_i}\right|_{|x_i| \to \infty} = 0 \quad (i = 1, 2). \tag{10}$$

Since the thermal-conductivity problem (7)-(10) is symmetrical relative to the midsection of the plane forming the plate, the differential equation for the determination of the Kirchhoff variable turns out to be a special case of the familiar [2] equation for an anisotropic plate and has the form

$$\frac{\partial^2 \vartheta}{\partial x_1^2} + \frac{\partial^2 \vartheta}{\partial x_2^2} - \beta^2 \vartheta = -\beta^2 Q_c, \tag{11}$$

where

$$\vartheta(x_1, x_2) = \frac{1}{2\delta} \int_{-\delta}^{\delta} \vartheta(x_1, x_2, x_3)\, dx_3, \quad \beta^2 = \frac{4\sigma_3}{\varkappa\delta}, \quad Q_c = \frac{q_c}{\beta^2 \delta \varkappa}.$$

Under boundary conditions (9) and (10), ϑ is a function of x_1, x_2.

Introducing the substitution $T = \vartheta - Q_c$, we come to the boundary-value problem

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} = \beta^2 T = 0, \tag{12}$$

$$\varkappa \frac{\partial T}{\partial x_i} = \mp q_0 \text{ when } x_i = \pm a_i, \ |x_{i\pm 1}| < a_{i\pm 1} \quad (i = 1, 2), \tag{13}$$

$$T|_{|x_i| \to \infty} = 0, \quad \left.\frac{\partial T}{\partial x_i}\right|_{|x_i| \to \infty} = 0 \quad (i = 1, 2). \tag{14}$$

Problem (12)-(14) is dealt with in [3]. With the method of extending the unknown function $T(x_1, x_2)$ through zero over the entire plane, the problem is reduced to the solution of a differential equation with singular coefficients which take into consideration the boundary condition at the outline of the rectangle. The unknown values of the function T at the boundary have been expanded into Fourier series. The solution of the equation is found through a convolution of the fundamental solution of the Helmholtz equation and of the right-hand side of this equation in the form

$$\begin{aligned}\Theta(x_1, x_2) = \frac{1}{2\pi} \sum_{i=1}^{2} \int_{x_{i\pm 1} - a_{i\pm 1}}^{x_{i\pm 1} + a_{i\pm 1}} \Bigg\{ \frac{q_0}{\varkappa} [K_0(\beta r_i^+) + K_0(\beta r_1^-)] - \\ - \beta \sum_{n=0}^{\infty} b_n^{(i)} \cos[\lambda_n^{(i\pm 1)} (\xi - x_{i\pm 1})] \left[\frac{x_i + a_i}{r_i^+} K_1(\beta r_i^+) - \right. \\ \left. - \frac{x_i - a_i}{r_i^-} K_1(\beta r_i^-) \right] \Bigg\} d\xi,\end{aligned} \tag{15}$$

where

$$\Theta(x_1, x_2) = T(x_1, x_2)\, M(x_1, x_2), \quad M(x_1, x_2) = 1 - M(x_1)\, M(x_2),$$

$$M(x_i) = S_+(x_i + a_i) - S_-(x_i - a_i), \quad r_i^{\pm} = \sqrt{(x_i \pm a_i)^2 + \xi^2}, \quad \lambda_n^{(i)} = \frac{\pi n}{a_i},$$

$$\Theta|_{x_i = a_i + 0}\, M(x_{i\pm 1}) = \sum_{n=0}^{\infty} b_n^{(i)} \cos(\lambda_n^{(i\pm 1)} x_{i\pm 1})\, M(x_{i\pm 1}).$$

The unknown Fourier coefficients $b_n^{(i)}$ in solution (15) have been found from such an infinite system of linear algebraic equations

$$b_k^{(i)} + \sum_{n=0}^{\infty} A_{kn}^{(i,i)} b_n^{(i)} = D_k^{(i)} \quad (i = 1,\ 2;\ k = 0,\ 1,\ \ldots), \tag{16}$$

where

$$D_k^{(i)} = \sum_{n=0}^{\infty} A_{kn}^{(i,i\pm 1)} b_n^{(i\pm 1)} + B_k^{(i)};$$

$$A_{kn}^{(i,i)} = \frac{1}{\pi a_{i\pm 1}} \int_0^{2a_{i\pm 1}} f_1(\xi,\ a_{i\pm 1},\ \lambda_k^{(i\pm 1)},\ \lambda_n^{(i\pm 1)})\, g(\xi,\ 2a_i)\, d\xi;$$

$$A_{kn}^{(i,i\pm 1)} = \frac{2}{\pi a_{i\pm 1}} \varepsilon(k)(-1)^{n+k+1} \int_0^{2a_{i\pm 1}} \cos \lambda_k^{(i\pm 1)} \xi \int_0^{2a_i} \cos \lambda_n^{(i)} \zeta g(\zeta,\ \xi)\, d\zeta d\xi;$$

$$B_k^{(i)} = \frac{q_0}{\pi \varkappa a_{i\pm 1}} \int_0^{2a_{i\pm 1}} \{(-1)^k \varepsilon(k) \cos \lambda_k^{(i\pm 1)} \xi \int_0^{2a_i} K_0(\beta \sqrt{\xi^2+\zeta^2})\, d\zeta -$$

$$- f_2(\xi,\ a_{i\pm 1},\ \lambda_k^{(i\pm 1)}) [K_0(\beta \sqrt{4a_i^2+\xi^2}) + K_0(\beta\xi)]\}\, d\xi;$$

$$f_1(\xi,\ b,\ \lambda_k,\ \lambda_n) = \begin{cases} 2b - \xi, & k = n = 0, \\ (2b-\xi)\cos\lambda_k\xi - \dfrac{1}{\lambda_k}\sin\lambda_k\xi, & k = n = 1,\ 2,\ \ldots; \\ \dfrac{(-1)^{k+n}}{\lambda_n^2-\lambda_k^2} 2\varepsilon(k)(\lambda_k \sin\lambda_k\xi - & k \neq n, \\ \qquad - \lambda_n \sin\lambda_n\xi), & k,\ n = 0,\ 1,\ \ldots; \end{cases}$$

$$f_2(\xi,\ b,\ \lambda_k) = \begin{cases} 2b - \xi, & k = 0; \\ \dfrac{2(-1)^k}{\lambda_k}\sin\lambda_k\xi, & k = 1,\ 2,\ \ldots; \end{cases} \quad \varepsilon(k) = \begin{cases} 0{,}5, & k = 0; \\ 1 & k = 1,\ 2,\ \ldots; \end{cases}$$

$$g(\xi,\ \zeta) = \frac{\beta\zeta}{\sqrt{\xi^2+\zeta^2}} K_1(\beta\sqrt{\xi^2+\zeta^2}).$$

Let us prove that system (16) has a solution which converges along the space norm ℓ^2, i.e.,

$$\sum_{n=0}^{\infty} [b_n^{(i)}]^2 < \infty \quad (i = 1,\ 2), \tag{17}$$

And for its solution we can use the method of reduction. For this we estimate the integrals in terms of which the coefficients of the system are expressed.

Since for $\xi > 0$ MacDonald function $K_0(\xi)$ is positive and diminishes monotonically, then according to [4]

$$0 \leqslant I_{1i}(k) = \int_0^{2a_{i\pm 1}} \sin(\lambda_k^{(i\pm 1)}\xi) K_0(\beta\xi)\, d\xi < \int_0^{\infty} \sin(\lambda_k^{(i\pm 1)}\xi) K_0(\beta\xi)\, d\xi$$

$$= \operatorname{arsh}\left(\frac{\lambda_k^{(i\pm 1)}}{\beta}\right) [\beta^2 + (\lambda_k^{(i\pm 1)})^2]^{-1/2}.$$

Consequently:

$$I_{1i}(k) < \frac{\text{const} \ln k}{k}. \tag{18}$$

Integrating the integral by parts

$$I_{2i}(k) = \int_0^{2a_{i\pm 1}} \sin(\lambda_k^{(i\pm 1)}\xi) K_0(\beta\sqrt{\xi^2+4a_i^2})\, d\xi,$$

we obtain

$$I_{2i}(k)=\frac{1}{\lambda_k^{(i\pm1)}}\,[K_0(2\beta a_i)-K_0(2\beta\sqrt{a_1^2+a_2^2})-$$

$$-\beta\int_0^{2a_{i\pm1}}\cos(\lambda_k^{(i\pm1)}\xi)\,g(\xi,\ 2a_i)\,d\xi]. \tag{19}$$

Since the function $g(\xi,\ 2a_i)$ is absolutely integrable in the interval $[0,\ 2a_{i\pm1}]$, the last term in Eq. (19) vanishes [5]. Then for $I_{2i}(k)$ we have the estimate

$$I_{2i}(k)=\frac{\text{const}}{k}+o\left(\frac{1}{k}\right). \tag{20}$$

In order to obtain an estimate of the integral

$$I_{3i}(n,\ k)=\int_0^{2a_{i\pm1}}\cos\lambda_k^{(i\pm1)}\xi\int_0^{2a_i}\cos\lambda_n^{(i)}\zeta g(\zeta,\ \xi)\,d\zeta d\xi,$$

we will use the integral representation [6]

$$\frac{\beta\xi}{\sqrt{\xi^2+\zeta^2}}K_1(\beta\sqrt{\xi^2+\zeta^2})=\int_0^\infty\frac{\eta\sin\eta\xi}{\sqrt{\beta^2+\eta^2}}\exp(-\zeta\sqrt{\beta^2+\eta^2})\,d\eta.$$

After a number of transformations we will obtain

$$I_{3i}(n,\ k)=\frac{2}{\beta^2+(\lambda_k^{(i\pm1)})^2+(\lambda_n^{(i)})^2}\int_0^\infty\left[\frac{\beta^2+(\lambda_k^{(i\pm1)})^2}{\beta^2+(\lambda_k^{(i\pm1)})^2+\eta^2}+\right.$$

$$\left.+\frac{\lambda_n^{(i)}}{\eta^2-(\lambda_k^{(i)})^2}\right]\sin^2\eta a_i\,[1-\exp(-2a_{i\pm1}\sqrt{\beta^2+\eta^2})]\,d\eta. \tag{21}$$

Let us examine each of the terms in (21). For the first term we have [6]

$$0\leqslant\int_0^\infty\frac{\sin^2\eta a_i}{\eta^2+\beta^2+(\lambda_k^{(i\pm1)})^2}\,[1-\exp(-2a_{i\pm1}\sqrt{\beta^2+\eta^2})]\,d\eta\leqslant$$

$$\leqslant\int_0^\infty\frac{\sin^2\eta a_i d\eta}{\eta^2+\beta^2+(\lambda_k^{(i\pm1)})^2}=\frac{\pi}{4}\,[\beta^2+(\lambda_k^{(i\pm1)})^2]^{-1/2}\,(1-\exp(-2a_i\sqrt{\beta^2+(\lambda_k^{(i\pm1)})^2})). \tag{22}$$

The second term in (21) can be written in the following form:

$$(\lambda_n^{(i)})^2\int_0^\infty\frac{\sin^2\eta a_i}{\eta^2-(\lambda_n^{(i)})^2}\,[1-\exp(-2a_{i\pm1}\sqrt{\beta^2+\eta^2})]\,d\eta=\frac{\beta}{2}K_1(2\beta a_{j\pm1})-$$

$$-a_{i\pm1}\beta\frac{K_1(2\beta\sqrt{a_1^2+a_2^2})}{2\sqrt{a_1^2+a_2^2}}+\frac{1}{2}\int_0^{2a_i}\cos\lambda_n^{(i)}\xi\frac{d}{d\xi}g(\xi,\ 2a_{i\pm1})\,d\xi. \tag{23}$$

The following estimate for the integral follows out of (22) and (23):

$$I_{3i}(n,\ k)=\frac{C_1n+C_2}{n^2+k^2}+o\left(\frac{1}{n^2+k^2}\right)\quad(C_i=\text{const}). \tag{24}$$

Integrating the integral by parts several times

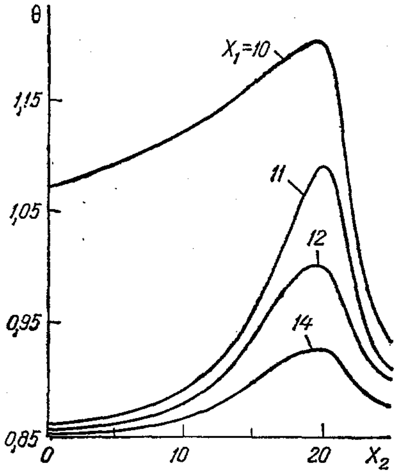

Fig. 1

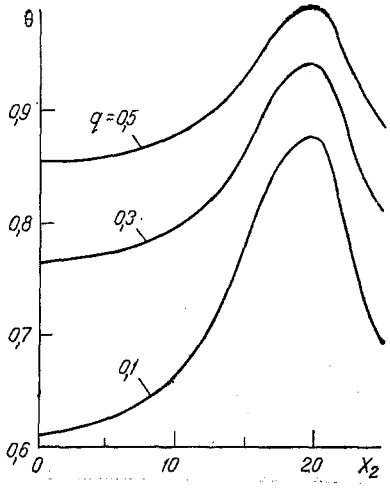

Fig. 2

Fig. 1. The dimensionless temperature θ as a function of $X_2 = x_2/\delta$ when X_1 = 10, 11, 12, 14; $q = q_c/q_0$ = 0.5.

Fig. 2. The dimensionless temperature θ as a function of X_2 when X_1 = 12; q = 0.1, 0.3, 0.5.

$$I_{4i}(n, k) = \frac{1}{(\lambda_n^{(i\pm1)})^2 - (\lambda_k^{(i\pm1)})^2} \int_0^{2a_{i\pm1}} [\lambda_k^{(i\pm1)} \sin \lambda_k^{(i\pm1)} \xi - \lambda_n^{(i\pm1)} \times$$
$$\times \sin \lambda_n^{(i\pm1)} \xi] \, g(\xi, 2a_i) \, d\xi \quad (n \neq k)$$

and taking into consideration the properties of the function under the integral sign, we obtain the following estimate:

$$I_{4i}(n, k) = \frac{\text{const}}{n^2 k^2} + o\left(\frac{1}{n^2 k^2}\right). \tag{25}$$

Analogously, through integration by parts for the integral

$$I_{5i}(k) = \int_0^{2a_{i\pm1}} \left[(2a_{i\pm1} - \xi) \cos \lambda_k^{(i\pm1)} \xi - \frac{1}{\lambda_k^{(i\pm1)}} \sin \lambda_k^{(i\pm1)} \xi \right] g(\xi, 2a_i) d\xi$$

we will have

$$I_{5i}(k) = \frac{\text{const}}{k^4} + o\left(\frac{1}{k^4}\right). \tag{26}$$

In order to estimate the integral

$$I_{6i}(k) = \int_0^{2a_{i\pm1}} \cos \lambda_k^{(i\pm1)} \xi \int_0^{2a_i} K_0(\beta \sqrt{\xi^2 + \zeta^2}) \, d\zeta d\xi,$$

we will use the following integral representation [6]:

$$K_0(\beta \sqrt{\xi^2 + \zeta^2}) = \int_0^\infty \frac{\cos \eta\xi}{\sqrt{\eta^2 + \beta^2}} \exp(-\zeta \sqrt{\eta^2 + \beta^2}) \, d\eta.$$

As a result we will obtain

$$I_{6i}(k) = \frac{1}{\beta^2 + (\lambda_k^{(i\pm1)})^2} \left\{ \frac{\pi}{2} [1 - \exp(-2a_{i\pm1}\beta)] - \right.$$

$$-\int_0^{2a_{i\pm1}} \cos\lambda_k^{(i\pm1)} \xi g(\xi,\ 2a_i)\, d\xi + \int_0^{\infty} \frac{\eta \sin 2\eta a_{i\pm1}}{\eta^2+\beta^2} \exp(-2a_i \sqrt{\beta^2+\eta^2})\, d\eta \Big\}.$$

From the absolute integrability of the function $K_1(\beta\sqrt{4a_i^2+\xi^2})/\sqrt{4a_i^2+\xi^2}$ in the interval $[0, 2a_{i\pm1}]$ it follows that the integral $\int_0^{2a_{i\pm1}} \cos\lambda_k^{(i\pm1)} \xi g(\xi,\ 2a_i)\, d\xi$ vanishes as $k \to \infty$. Therefore, for $I_{6i}(k)$ we have the estimate

$$I_{6i}(k) = \frac{\text{const}}{k^2} + o\left(\frac{1}{k^2}\right). \tag{27}$$

Using estimates (18), (20), (24)-(27), for the coefficients of system (16) we obtain the following estimates:

$$\begin{gathered} A_{0n}^{(i,i)} = \frac{L_{1i}}{n^2} + o\left(\frac{1}{n^2}\right),\quad A_{0n}^{(i,i\pm1)} = \frac{L_{2i}}{n} + o\left(\frac{1}{n}\right) \quad (n = 1,\ 2,\ \ldots), \\ A_{kn}^{(i,i)} = \frac{L_{3i}}{n^2k^2} + o\left(\frac{1}{n^2k^2}\right),\quad A_{kn}^{(i,i\pm1)} = \frac{L_{4i}n + L_{5i}}{n^2+k^2} + o\left(\frac{1}{n^2+k^2}\right) \\ (k,\ n = 1,\ 2,\ \ldots), \\ B_k^{(i)} < \frac{L_{6i}\ln k}{k} \quad (k = 1,\ 2,\ \ldots),\quad L_{pi} = \text{const}. \end{gathered} \tag{28}$$

From estimates (28) and condition (17) it follows that

$$\sum_{k,n=0}^{\infty} [A_{kn}^{(i,i)}]^2 < \infty,\quad \sum_{k=0}^{\infty} [D_k^{(i)}]^2 < \infty \quad (i = 1,\ 2). \tag{29}$$

The derived estimates enable us to apply the theory of infinite-system solubility to system (16) [7]. The approximate solution of the system can be obtained by the reduction method.

Let us undertake a numerical analysis of the derived results for $A_1 = a_1/\delta = 10$, $A_2 = a_2/\delta = 20$, $Bi = \beta^2\delta^2 = 0.1$, using formulas (15), (5), and (6).

Figure 1 graphs of the dimensionless temperature $\theta = t\sqrt[4]{\kappa/q_0\delta}$ as a function of the dimensionless coordinate $X_2 = x_2/\delta$ for the case in which $q = q_c/q_0 = 0.5$ in the vicinity of the notch and at its boundary. We can see from the graphs that the maximum value of the temperature is attained at the break point.

The dimensionless temperature θ as a function of X_2 for the case in which $X_1 = x_1/\delta = 12$, $q = 0.1, 0.3, 0.5$ is shown in Fig. 2. From these functions we see that the temperature increases as q increases.

NOTATION

t, temperature field; ϑ, Kirchhoff variables; 2δ, plate thickness; $2a_1 \times 2a_2$, notch dimensions; x_1, x_2, x_3, rectangular Cartesian coordinates; λ, coefficient of thermal conductivity; q_c, heat flow from emitter at the surfaces $x_3 = \pm\delta$; σ_3, visible coefficient of radiative heat exchange; q_0, heat flow at the rectangular boundaries of the plate; $S_\pm(\xi)$, asymmetric unit functions; $K_\nu(\xi)$, MacDonald function of order ν.

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